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Periodic Solutions of Conservation Laws

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1. INTRODUCTION

Let $\phi = (\phi_1, ..., \phi_m)$: $\mathbb{R} \to \mathbb{R}^m$ be a continuously differentiable function, and let $p = (p_1, ..., p_m)$ be an m-tuple of positive numbers. Of concern is the following periodic initial value problem for a single conservation law in m variables:

$$\frac{\partial u}{\partial t} + \sum_{i=1}^{m} \frac{\partial}{\partial x_i}(\phi_i(u)) = 0 \qquad (t \geqslant 0,\ x \in \mathbb{R}^m), \tag{1}$$

$$u(0, x) = u_0(x) \qquad (x \in \mathbb{R}^m), \tag{2}$$

$$u(t, x + p) = u(t, x) \qquad (t \geqslant 0,\ x \in \mathbb{R}^m). \tag{3}$$

In 1972, M. G. Crandall [4] solved (1), (2) by associating with this problem a contraction semigroup on $L^1(\mathbb{R}^m)$. Here we prove an analogous result for (1), (2), (3).

DEFINITION 1. Let X be the Banach space of (equivalence classes of) locally integrable, p-periodic functions on $\mathbb{R}^m$ with the norm $\| f \| = \int_{Q_p} |f(x)|\, dx$, where $Q_p = \{(x_1, ..., x_m) \in \mathbb{R}^m: 0 < x_i < p_i \text{ for } 1 \leqslant i \leqslant m\}$. If $v \in X \cap L^\infty(\mathbb{R}^m)$, then we will say that v is in the domain of A_0 and $A_0 v = w$ provided that $w \in X$, v is the a.e. limit of a sequence of continuous functions, and

$$\int_{Q_p} \operatorname{sign}_0(v(x) - k) \left[\sum_{i=1}^{m} (\phi_i(v(x)) - \phi_i(k)) \frac{\partial f}{\partial x_i}(x) - w(x) f(x) \right] dx \geqslant 0$$

for all nonnegative $f \in C_0^\infty(Q_p)$ and all $k \in \mathbb{R}$; here $\operatorname{sign}_0(s) = s/|s|$ for $s \neq 0$ and $\operatorname{sign}_0(0) = 0$. (Necessarily $w = -\sum_{i=1}^{m} (\partial/\partial x_i)(\phi_i(v))$ in the sense of distributions, but an "entropy condition" is built into this definition, cf. [4].)

THEOREM. *The closure A of A_0 is densely defined and m-dissipative on X; hence (by [5]) it generates a strongly continuous contraction semigroup $\{T(t);\ t \geqslant 0\}$ of operators on X. If $u(t) = T(t)\, u_0$ for $u_0 \in X$, $t \geqslant 0$, then u is the unique solution of* (1)–(3) *in the sense of Benilan* [1], *in the sense of Kružkov* [13].

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For motivation and background material on conservation laws, see e.g. Lax [15]. The discovery that L^1-norm is the right norm for semigroup purposes was made by Quinn [17].

We remark that our definition of the domain of A_0 is more restrictive than the definition given by Crandall [4]. The proof of the dissipativity, which is patterned after Crandall's proof, requires the construction of Kružkov type test functions having special periodicity and symmetry properties. The proof of the "m" part (or range condition) is more complicated than Crandall's because of the extra requirements for a function to be in the domain of A^0.

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2. Notation

Let $Q_p = \{(x_1, \ldots, x_m) \in \mathbb{R}^m\colon\ 0 < x_i < p_i,\, i = 1, \ldots, m\}$ for $0 < p_i \in \mathbb{R}$, $i = 1, \ldots, m$. Hereafter $p = (p_1, \ldots, p_m) \in \mathbb{R}^m$ is fixed. Let $\phi = (\phi_1, \ldots, \phi_m)$: $\mathbb{R} \to \mathbb{R}^m$ satisfy $\phi(0) = 0$ and ϕ is continuously differentiable. Let u: $\mathbb{R}^m \to \mathbb{R}$, and let $u_x = (u_{x_1}, \ldots, u_{x_m})$ be the gradient of u. Let $\phi' = (\phi_1', \ldots, \phi_m')$, $\phi(u)_x = \phi'(u) \cdot u_x = \sum_{i=1}^m \phi_i'(u)\, u_{x_i}$ where $\cdot$ is the Euclidean dot product. We will write $\|\cdot\|_q$, $\|\cdot\|_{H^2}$ for $\|\cdot\|_{L^q(Q_p)}$, $\|\cdot\|_{H^2(Q_p)}$ respectively.

We regard the Lebesgue space $L^q(Q_p)$ and the Sobolev space $H^2(Q_p)$ as real Banach spaces. Let $C_p^\infty \equiv C_{Q_p}^\infty(\mathbb{R}^m) = \{g \in C^\infty(\mathbb{R}^m)\colon g(x + \hat{p}_i) = g(x)$, for all $x \in \mathbb{R}^m$, $1 \leqslant i \leqslant m\}$, where $\hat{p}_i = (0, \ldots, 0, p_i, 0, \ldots, 0)$, p_i being the i-th place. Next, set $C_0^\infty[Q_p] = \{f \in C_p^\infty\colon f|_{Q_p} \in C_0^\infty(Q_p)\}$, the subscript 0 standing for compact support.

Our basic Banach space will be $X = \{f\colon \mathbb{R}^m \to \mathbb{R}\colon f$ is measurable $f(x + \hat{p}_i) = f(x)$, for all $x \in \mathbb{R}^m$, $i = 1, \ldots, m$, $f|_{Q_p} \in$ closure of C_p^∞ in $L^1(Q_p)$ norm$\}$ equipped with the $L^1(Q_p)$ norm. X is a real, separable, nonreflexive Banach space, isomorphic to $L^1(Q_p)$. The norm in X will be denoted by $\|\cdot\|_1$.

3. Preliminary Lemmas

Remark 1. A_0 is a single-valued function and for $v \in D(A_0)$, $A_0 v = -(\phi(v))_x$ in the sense of distributions.

Proof. Letting $k \geqslant \|v\|_\infty$ in (1), we get

$$-\int_{Q_p} [\phi(v(x)) \cdot f_x(x) - w(x)\, f(x)]\, dx \geqslant 0$$

for all $f \in C_0^\infty[Q_p]$, $f \geqslant 0$, since $\int_{Q_p} \phi(k) \cdot f_x\, dx = \sum_{i=1}^m \int_{Q_i'} \phi_i(k)\, (f(\bar{p}_i) - f(\bar{0}_i))\, dx_i' = 0$ where $Q_i' = \prod_{j \neq i} (0, p_k)$, $dx_i' = dx_1 \cdots dx_{i-1}\, dx_{i+1} \cdots dx_m$, $\bar{p}_i = (x_1, \ldots, x_{i-1},$

$p_i, x_{i+1}, \ldots, x_m)$, $\bar{0}_i = (x_1, \ldots, x_{i-1}, 0, x_{i+1}, \ldots, x_m)$. Letting $k < -\| v \|_\infty$ in (1), we get similarly

$$\int_{Q_p} [\phi(v(x)) \cdot f_x(x) - w(x) f(x)]\, dx \geqslant 0$$

for all $f \in C_0^\infty[Q_p]$, $f \geqslant 0$. Hence

$$\int_{Q_p} [\phi(v(x)) \cdot f_x(x) - w(x) f(x)]\, dx = 0$$

for all $f \in C_0^\infty[Q_p]$, $f \geqslant 0$.

Since the linear span of the set $\{f \in C_0^\infty[Q_p] : f \geqslant 0\}$ is dense in $C_0^\infty[Q_p]$, hence we get by a standard passage to the limit,

$$\int_{Q_p} [\phi(v(x)) \cdot f_x(x) - w(x) f(x)]\, dx = 0$$

for all $f \in C_0^\infty[Q_p]$, thus $w = -(\phi(v(x)))_x$ in the distribution sense. ∎

Definition 2. Let $u: \mathbb{R}^m \to \mathbb{R}$ be measurable. Then $\operatorname{sgn} u = \{s: \mathbb{R}^m \to \mathbb{R}:$ s is measurable, $|s(x)| \leqslant 1$ a.e. and $s(x)\, u(x) = |u(x)|$ a.e.$\}$.

Lemma 1. *Let $u \in X$, $v \in X$, $\alpha \in \operatorname{sgn} u$. If $\int_{Q_p} \alpha v \, dx \geqslant 0$, then*

$$\| u + \lambda v \|_1 \geqslant \| u \|_1 \qquad \textit{for all } \lambda \geqslant 0.$$

Proof. Since $| u + \lambda v | \geqslant | \alpha | \, | u + \lambda v | = | \alpha(u + \lambda v)| \geqslant \alpha(u + \lambda v) = | u | + \lambda \alpha v$, a.e. in Q_p, hence $\| u + \lambda v \|_1 = \int_{Q_p} | u + \lambda v | \, dx \geqslant \int_{Q_p} | u | \, dx + \lambda \int_{Q_p} \alpha v \, dx \geqslant \int_{Q_p} | u | \, dx = \| u \|_1$. ∎

4. The Dissipative Property

Theorem 1. *Let A_0 be given by Definition 1. Then A_0 is dissipative.*

Proof. The proof is long and will be done in a number of parts. Let $u, v \in D(A_0)$, $w = A_0 v$, $z = A_0 u$. What we must show is that $\| v - u - \lambda(w - z)\|_1 \geqslant \| u - v \|_1$ for all $\lambda > 0$.

Let $g(x, y) \in C_0^\infty(Q_p \times Q_p)$, and extend it to $\mathbb{R}^m \times \mathbb{R}^m$ by periodicity. Set $k = u(y)$, $f(x) = g(x, y)$ in (1) and then integrate this over Q_p with respect to y; we get

$$\int_{Q_p \times Q_p} \operatorname{sign}_0(v(x) - u(y)) \, [(\phi(v(x)) - \phi(u(y))) \cdot g_x(x, y) - w(x)\, g(x, y)] \, dx\, dy \geqslant 0. \tag{2}$$

Interchange u, v and z, w and x, y, we get another inequality symmetric to (2); adding these two gives us

$$\int_{Q_p \times Q_p} \operatorname{sign}_0(v(x) - u(y))\,(\phi(v(x)) - \phi(u(y))) \cdot (g_x + g_y) - (w(x) - z(y))g)\,dx\,dy \geqslant 0. \tag{3}$$

By periodicity of the integrand, we get from (3)

$$\int_{R_p} \operatorname{sign}_0(v(x) - u(y))\,(\phi(v(x)) - \phi(u(y))) \cdot (g_x + g_y) - (w(x) - z(y))\,g)\,dx\,dy \geqslant 0$$

where $R_p = \prod_{i=1}^m R_{p_i}$ and R_{p_i} is the bounded region in the $x_i - y_i$ plane bounded by the four lines $x_i = \pm y_i$, $x_i = 2p_i \pm y_i$; it has the property that under the coordinate transformation $(x, y) \mapsto ((x + y)/2, (x - y)/2)$, we have $R_p \to Q_p \times Q_p$, and the Jacobian is one.

Now let $\delta^k \in X$ satisfy the following conditions: $\delta^k \in C_0^\infty[Q_p]$ is non-negative, $\int_{Q_p} \delta^k = 1$, $k = 1, 2,\ldots$ and $\operatorname{supp} \delta^k |_{Q_p} \subset \prod_{i=1}^m [\epsilon_1{}^k, \epsilon_2{}^k]$ with $0 < \epsilon_1{}^k < \epsilon_2{}^k \to 0$ as $k \to \infty$. (Thus $\delta^k |_{Q_p}$ is supported in a small box close to the origin.) Let $f \in C_0^\infty[Q_p]$, $f \geqslant 0$ and set $g(x, y) = f((x + y)/2)\,\delta^k((x - y)/2)$ in (3). Let $2\xi = x + y$, $2\eta = x - y$ in this result to obtain

$$\int_{Q_p} \left[\int_{Q_p} \operatorname{sign}_0(v(\xi + \eta) - u(\xi - \eta))\,\{(\phi(v(\xi + \eta)) - \phi(u(\xi - \eta))) \cdot f_\xi(\xi) - (w(\xi + \eta) - z(\xi - \eta))\,f(\eta)\}\,d\xi\right] \delta^k(\eta)\,d\eta \geqslant 0.$$

Let $I_f(\eta)$ denote the integral in $[\cdots]$ in the above inequality. Since $f \in C_0^\infty[Q_p]$, $I_f(\eta)$ is bounded, and we have

$$0 \leqslant \liminf_{k\to\infty} \int_{Q_p} I_f(\eta)\,\delta^k(\eta)\,d\eta \leqslant \limsup_{|\eta|\to 0} I_f(\eta).$$

Now choose a sequence $\{\eta_k\} \subset \mathbb{R}^m$ such that $|\eta_k| \to 0$ and $\lim_{k\to\infty} I_f(\eta_k) = \limsup_{|\eta|\to 0} I_f(\eta)$. Let $\alpha_k(\xi) = \operatorname{sign}_0(v(\xi + \eta_k) - u(\xi - \eta_k)) \in \operatorname{sgn}(v(\xi + \eta_k) - u(\xi - \eta_k))$.

LEMMA 2. *Let X_0 be a real separable Banach space and X_0^* its dual. Let $H(x) = \{x^* \in X_0^* : (x, x^*) = \| x \|$ and $\| x^* \| \leqslant 1\}$, where (x, x^*) denotes the value of x^* at x. Let $\{x_k\} \subseteq X_0$ and $x_k^* \in H(x_k)$ for $k = 1, 2,\ldots$. If $\{x_k\}$ converges strongly in X_0 to x, then $\{x_k^*\}$ has a subsequence $\{x_{n_k}^*\}$ convergent weakly-star to an element x^* of $H(x)$.*

Proof. See Lemma 4.2 of [4]. ∎

Set $X_0 = X$, $x_k = v(\cdot + \eta_k) - u(\cdot - \eta_k)$ in the above lemma; we get that $\alpha_k\ (= x_k^*)$ has a weakly-star convergent subsequence, still denotes by α_k for typographical convenience, convergent to an element $\alpha \in \operatorname{sgn}(v - u) = H(v - u)$. Since

$$f_\xi(\xi) \cdot (\phi(v(\xi + \eta)) - \phi(u(\xi - \eta))) + f(\xi)\,(w(\xi + \eta) - z(\xi - \eta)) \in L^1(Q_p),$$

considered as a function of ξ with η fixed, hence

$$\begin{aligned} 0 &\leqslant \limsup_{\eta \to 0} I_f(\eta) = \lim_{k\to\infty} I_f(\eta_k) \\ &= \int_{Q_p} \alpha(\xi)\,[(\phi(v(\xi)) - \phi(u(\xi))) \cdot f_\xi(\xi) + (w(\xi) - z(\xi))\,f(\xi)]\,d\xi. \end{aligned} \tag{4}$$

LEMMA 3. *Let u, v, ϕ, α be as above and let $\epsilon > 0$. Then there exists an $f \in C_0^\infty[Q_p]$ such that*

$$\left| \int_{Q_p} \alpha(\xi)\,(\phi(v(\xi)) - \phi(u(\xi))) \cdot f_\xi(\xi)\,d\xi \right| < \epsilon.$$

Proof. For given $\epsilon > 0$ let u_j, v_j be continuous functions converging to u, v a.e. as $j \to \infty$ respectively. (Cf. Definition 1.) Then by Egorov's theorem (see [11]), there is an $A \subset Q_p$ of arbitrarily small Lebesgue measure (depending on ϵ) such that $u_j \to u$, $v_j \to v$ uniformly on $B \equiv Q_p \backslash A$; hence u, v are continuous on B. (How large the measure $\mu(A)$ should be will be specified later when we symmetrize A.) Moreover, we can choose x_0 and δ (depending on x_0) such that the following two conditions are satisfied. First,

$$\mu(P_i(A \cap C^i)) < \frac{\epsilon}{12\beta m} \qquad \text{for } i = 1,\dots, m \tag{5}$$

where P_i is the projection $(\xi_1, \dots, \xi_m) \mapsto (\xi_1, \dots, \xi_{i-1}, \xi_{i+1}, \dots, \xi_m)$, $C^i = (0, p_1) \times \cdots \times (0, p_{i-1}) \times \{(x_0 - \delta, x_0 + \delta) \cup (p_i - x_0 - \delta, p_i - x_0 + \delta)\} \times (0, p_{i+1}) \times \cdots \times (0, p_m)$, $\beta > \|\phi_i(v) - \phi_i(u)\|_\infty$ (for $i = 1,\dots, m$) is fixed, and μ is the Lebesgue measure on $\mathbb{R}^{m-1}$. Secondly,

$$\begin{aligned} |\phi_i(u(\xi)) - \phi_i(u(\bar{x}_0^{\,i}))| &< \frac{\epsilon}{24m \prod_{j\neq i} p_j}, \\ |\phi_i(v(\xi)) - \phi_i(v(\bar{x}_0^{\,i}))| &< \frac{\epsilon}{24m \prod_{j\neq i} p_j} \end{aligned} \tag{6}$$

for $i = 1,\dots, m$ where $\bar{x}_0^{\,i} = (\xi_1, \dots, \xi_{i-1}, x_0, \xi_{i+1}, \dots, \xi_m)$ and $|\xi - \bar{x}_0^{\,i}| < \delta$ and ξ, $\bar{x}_0^{\,i} \in B$.

By enlarging B (and diminishing A) if necessary, we may suppose that the Lebesgue measure of A satisfies $\mu(A) < \epsilon/12\beta m$, and moreover, A is symmetric with respect to both $\bar{x}_0^i$ and $(\xi_1, ..., \xi_{i-1}, p_i - x_0, \xi_{i+1}, ..., \xi_m)$ for each fixed $(\xi_1, ..., \xi_{i-1}, \xi_{i+1}, ..., \xi_m)$. Let $E_0^i = \{\hat{\xi}_i = (\xi_i, ..., \xi_{i-1}, \xi_{i+1}, ..., \xi_m): u(\bar{x}_0^i) = v(\bar{x}_0^i)$ or $u(\overline{p_i - x_0}) = v(\overline{p_i - x_0})$, where $\bar{x}_0^i = (\xi_1, ..., \xi_{i-1}, x_0, \xi_{i+1}, ..., \xi_m)$ and $\overline{p_i - x_0} = (\xi_1, ..., \xi_{i-1}, p_i - x_0, \xi_{i+1}, ..., \xi_m)\}$, and for $\gamma > 0$ let $E^i(\gamma)$ be the γ-neighborhood of the boundary ∂E_0^i of E_0^i in $P_i(B)$.

Then choose an $\epsilon_0 > 0$ (depending on ϵ) such that $\mu(E^i(\epsilon_0)) < \epsilon/12\beta m$. Now diminish δ (depending on ϵ_0) if necessary so that the sets E_1^i, $E_2^i \subset P_i(B)$ satisfy the following properties:

$$\begin{aligned} E_1^i &= \{\hat{\xi}_i \in P_i(B)\backslash E^i(\epsilon_0): \alpha(\overline{x_0 + \xi_i}) = \alpha(\overline{x_0 - \xi_i}) \neq 0 \\ &\qquad \text{and } \alpha(\overline{p_i - x_0 + \xi_i}) = \alpha(\overline{p_i - x_0 - \xi_i}) \neq 0, \text{ for } \xi_i \in [0, \delta]\}, \\ E_2^i &= \{\hat{\xi}_i \in P_i(B): \alpha(\bar{x}_0^i) = 0\}\backslash(E_1^i \cup E^i(\epsilon_0)) \\ &\qquad \text{and } E_1^i \cup E_2^i \cup E^i(\epsilon_0) = P_i(B) \text{ and } E_1^i, E_2^i, E^i(\epsilon_0) \end{aligned}$$

are mutually disjoint.

Let $f(\xi) = \prod_{i=1}^m f_i(\xi)$ where $f_i \in C_0^\infty(0, p_i)$,

$$\operatorname{supp} f_i \subset [x_0 - \delta, p_i - x_0 + \delta], f_i|_{[x_0+\delta, p_i-x_0-\delta]} \equiv 1,$$

$$\operatorname{supp} \frac{df_i}{d\xi_i} \subset [x_0 - \delta, x_0 + \delta] \cup [p_i - x_0 - \delta, p_i - x_0 + \delta],$$

and the derivatives $df_i/d\xi_i|_{[x_0-\delta, x_0+\delta]} \geqslant 0$, $df_i/d\xi_i|_{[p_i-x_0-\delta, p_i-x_0+\delta]} \leqslant 0$ are symmetric with respect to x_0, $p_i - x_0$ respectively. Then

$$\begin{aligned} &\left| \int_{Q_p} \alpha(\phi(v) - \phi(u)) \cdot f_\xi \, d\xi \right| \\ &\quad \leqslant \sum_{i=1}^m \left| \int_{Q_p} \alpha(\phi_i(v) - \phi_i(u)) f_{\xi_i} \, d\xi \right| \qquad (7) \\ &\quad \leqslant \sum_{i=1}^m \left(\left| \int_A \alpha(\phi_i(v) - \phi_i(u)) f_{\xi_i} \, d\xi \right| + \left| \int_B \alpha(\phi_i(v) - \phi_i(u)) f_{\xi_i} \, d\xi \right| \right). \end{aligned}$$

But, letting χ_A denote the characteristic (or indicator) function of the set A, we get

$$\begin{aligned} \int_{Q_p} \chi_A |f_{\xi_i}| \, d\xi &\leqslant \left(\int_{x_0-\delta}^{x_0+\delta} + \int_{p_i-x_0-\delta}^{p_i-x_0+\delta} \right) \left(\int_{Q_p'} \chi_A(\xi) \prod_{j\neq i} f_j(\xi_j) \, d\xi_i' \right) \left| \frac{df_i}{d\xi_i}(\xi_i) \right| d\xi_i \\ &\leqslant \frac{\epsilon}{12\beta m} \left(\int_{x_0-\delta}^{x_0+\delta} + \int_{p_i-x_0-\delta}^{p_i-x_0+\delta} \right) \left| \frac{df_i}{d\xi_i}(\xi_i) \right| d\xi_i = \frac{\epsilon}{6\beta m} \end{aligned}$$

where we have used (5) in the last inequality and the facts

$$\int_{x_0-\delta}^{x_0+\delta} \frac{df_i}{d\xi_i}(\xi_i)\,d\xi_i = 1, \qquad \int_{p_i-x_0-\delta}^{p_i-x_0+\delta} \frac{df_i}{d\xi_i}(\xi_i)\,d\xi_i = -1.$$

Thus we have

$$\begin{aligned} &\left| \int_A \alpha(\phi_i(v) - \phi_i(u)) f_{\xi_i}\,d\xi \right| \\ &\qquad \leqslant \int_{Q_p} |\,\alpha(\phi_i(v) - \phi_i(u))\,\chi_A f_{\xi_i}\,|\,d\xi \qquad (8)\\ &\qquad \leqslant \|\phi_i(v) - \phi_i(u)\|_\infty \int_{Q_p} \chi_A\,|\,f_{\xi_i}\,|\,d\xi \leqslant \|\phi_i(v) - \phi_i(u)\|_\infty \frac{\epsilon}{6m\beta} < \frac{\epsilon}{6m}. \end{aligned}$$

Next,

$$\begin{aligned} &\left| \int_B \alpha(\phi_i(v) - \phi_i(u)) f_{\xi_i}\,d\xi \right| \\ &\qquad = \left| \int_{P_i(B)} \left[\left(\int_{x_0-\delta}^{x_0+\delta} + \int_{p_i-x_0-\delta}^{p_i-x_0+\delta} \right) \alpha(\bar{\xi}_i)\,(\phi_i(v(\bar{\xi}_i)) - \phi_i(u(\bar{\xi}_i)))\,\frac{df_i}{d\xi_i}(\xi_i)\,d\xi_i \right] \right.\\ &\qquad\qquad \left. \times \prod_{j\neq i} f_j(\xi_j)\,d\xi_i' \right| \\ &\qquad \leqslant \left| \int_{E^i(\epsilon_0)} \left[\left(\int_{x_0-\delta}^{x_0+\delta} + \int_{p_i-x_0-\delta}^{p_i-x_0+\delta} \right) I_0\,d\xi_i \right] \prod_{j\neq i} f_j(\xi_j)\,d\xi_i' \right| \qquad (9)\\ &\qquad\qquad + \left| \int_{E_1^i} \left[\left(\int_{x_0-\delta}^{x_0+\delta} + \int_{p_i-x_0-\delta}^{p_i-x_0+\delta} \right) I_0\,d\xi_i \right] \prod_{j\neq i} f_j(\xi_j)\,d\xi_i' \right| \\ &\qquad\qquad + \left| \int_{E_2^i} \left[\left(\int_{x_0-\delta}^{x_0+\delta} + \int_{p_i-x_0-\delta}^{p_i-x_0+\delta} \right) I_0\,d\xi_i \right] \prod_{j\neq i} f_j(\xi_j)\,d\xi_i' \right| \equiv I_1 + I_2 + I_3, \end{aligned}$$

where $I_0 = \alpha(\bar{\xi}_i)\,(\phi_i(v(\bar{\xi}_i)) - \phi_i(u(\bar{\xi}_i)))\,df_i/d\xi_i\,(\xi_i)$, $d\xi_i' = d\xi_1 \cdots d\xi_{i-1}\,d\xi_{i+1} \cdots d\xi_m$, and $\bar{\xi}_i = (\xi_1, \dots, \xi_{i-1}, \xi_i, \xi_{i+1}, \dots, \xi_m)$ with $\hat{\xi}_i = (\xi_1, \dots, \xi_{i-1}, \xi_{i+1}, \dots, \xi_m)$ fixed.

Similar to the case of $|\int_A \alpha(\phi_i(v) - \phi_i(u)) f_{\xi_i}\,d\xi\,|$ we have

$$\int_{Q_p} \chi_{E^i(\epsilon_0)}\,|\,f_{\xi_i}\,|\,d\xi \leqslant \frac{\epsilon}{6\beta m}.$$

Thus we get

$$\begin{aligned} I_1 &= \left| \int_{E^i(\epsilon)} \alpha(\phi_i(v) - \phi_i(u))\,f_{\xi_i}\,d\xi \right| \\ &\leqslant \int_{Q_p} |\,\alpha(\phi_i(v) - \phi_i(u))\,\chi_{E^i(\epsilon_0)} f_{\xi_i}\,|\,d\xi \end{aligned}$$

$$\leqslant \|\phi_i(v) - \phi_i(u)\|_\infty \int_{Q_p} \chi_{E^i(\epsilon_0)} \, |f_{\xi_i}| \, d\xi$$

$$\leqslant \|\phi_i(v) - \phi_i(u)\|_\infty \frac{\epsilon}{6m\beta} < \frac{\epsilon}{6m}.$$

We now consider the cases of I_2 and I_3.

If $\hat{\xi}_i \in E_1{}^i$, then $\alpha(\overline{x_0 + \xi_i}) = \alpha(x_0 - \xi_i) \neq 0$ for $\xi_i \in [0, \delta]$, where $\overline{x_0 \pm \xi_i} = (\xi_1, \dots, \xi_{i-1}, x_0 \pm \xi_i, \xi_{i+1}, \dots, \xi_m)$ respectively with $\hat{\xi}_i$ fixed, and

$$\left| \int_{x_0-\delta}^{x_0+\delta} \alpha(\bar{\xi}_i) \, (\phi_i(v(\bar{\xi}_i)) - \phi_i(u(\bar{\xi}_i))) \frac{df_i}{d\xi_i}(\xi_i) \, \chi_B(\bar{\xi}_i) \, d\xi_i \right|$$

(by a change of variable and symmetry of $df_i/d\xi_i$ and B)

$$\begin{aligned}
&= \Big| \int_0^\delta \alpha(\overline{x_0 + \xi_i}) \, [(\phi_i(v(\overline{x_0 + \xi_i})) - \phi_i(u(\overline{x_0 + \xi_i})))\\
&\qquad - (\phi_i(v(\overline{x_0 - \xi_i})) - \phi_i(u(\overline{x_0 - \xi_i})))] \frac{df_i}{d\xi_i}(\overline{x_0 + \xi_i}) \, \chi_B(\overline{x_0 + \xi_i}) \, d\xi_i \Big|\\
&\leqslant \operatorname*{ess\,sup}_{\xi_i \in [0,\delta]} | \phi_i(v(\overline{x_0 + \xi_i})) - \phi_i(u(\overline{x_0 + \xi_i}))\\
&\qquad - (\phi_i(v(\overline{x_0 - \xi_i})) - \phi_i(u(\overline{x_0 - \xi_i})))|\\
&< \frac{\epsilon}{6m \prod_{j \neq i} p_j}
\end{aligned}$$

by (6). Thus we get

$$I_2 \leqslant \frac{\epsilon}{3m \prod_{j \neq i} p_j} \left| \int_{E_1{}^i} \prod_{j \neq i} f_j(\xi_j) \, d\xi_i' \right| \leqslant \frac{\epsilon \mu(E_1{}^i)}{3m \prod_{j \neq i} p_j}.$$

If $\hat{\xi}_i \in E_2{}^i$, then $\alpha(\bar{x}_0{}^i) = 0$ and

$$\begin{aligned}
&\left| \int_{x_0-\delta}^{x_0+\delta} \alpha(\bar{\xi}_i) \, (\phi_i(v(\bar{\xi}_i)) - \phi_i(u(\bar{\xi}_i))) \frac{df_i}{d\xi_i}(\xi_i) \, \chi_B(\bar{\xi}_i) \, d\xi_i \right|\\
&\qquad \leqslant \Big| \int_0^\delta \alpha(\overline{x_0 + \xi_i}) \, (\phi_i(v(\overline{x_0 + \xi_i}))\\
&\qquad\qquad - \phi_i(u(\overline{x_0 + \xi_i}))) \frac{df_i}{d\xi_i}(\overline{x_0 + \xi_i}) \, \chi_B(\overline{x_0 + \xi_i}) \, d\xi_i \Big|\\
&\qquad\qquad + \Big| \int_0^\delta \alpha(\overline{x_0 - \xi_i}) \, (\phi_i(v(\overline{x_0 - \xi_i}))\\
&\qquad\qquad - \phi_i(u(\overline{x_0 - \xi_i}))) \frac{df_i}{d\xi_i}(\overline{x_0 - \xi_i}) \, \chi_B(\overline{x_0 - \xi_i}) \, d\xi_i \Big| < \frac{\epsilon}{6m \prod_{j \neq i} p_j}
\end{aligned}$$

by (6). Hence we get

$$I_3 < \frac{\epsilon}{3m \prod_{j \neq i} p_j} \left| \int_{E_2{}^i} \prod_{j \neq i} f_j(\xi_j)\, d\xi_i' \right| < \frac{\epsilon \mu(E_2{}^i)}{3m \prod_{j \neq i} p_j}.$$

Thus from the above inequalities for I_1, I_2, I_3 and (9) together with the properties of f, we get

$$\begin{aligned} \left| \int_B \alpha(\phi_i(v) - \phi_i(u)) f_{\xi_i}\, d\xi \right| &\leqslant I_1 + I_2 + I_3 \\ &\leqslant \frac{\epsilon}{6m} + \frac{\epsilon \mu(E_1{}^i)}{3m \prod_{j \neq i} p_j} + \frac{\epsilon \mu(E_2{}^i)}{3m \prod_{j \neq i} p_j} \leqslant \frac{\epsilon}{2m}. \end{aligned} \tag{10}$$

Hence, by (7), (8), (10), we get Lemma 3. ∎

Next we choose a sequence $\{\epsilon_k\}$ of positive real numbers decreasing to zero and the corresponding sequence $\{f^k\} \subset C_0{}^\infty[Q_p]$ as in Lemma 3. Then by (4), we have $0 \leqslant \int_{Q_p} \alpha[(\phi(v) - \phi(u)) \cdot f_\xi{}^k - (w - z) f^k]\, d\xi$ for each k, and $\int_{Q_p} \alpha(\phi(v) - \phi(u)) \cdot f_\xi{}^k\, d\xi \to 0$ as $k \to \infty$ by Lemma 3, then letting $k \to \infty$, it follows easily that $\int_{Q_p} \alpha(\xi)\, (w(\xi) - z(\xi))\, d\xi \geqslant 0$. Then from Lemma 1, it follows that $\| v - u - \lambda(w - z)\|_1 \geqslant \| v - u \|_1$ for $\lambda \geqslant 0$. Since $u, v \in D(A_0)$ and $w \in A_0 v$, $z \in A_0 u$ were arbitrary, the proof of Theorem 1 is complete. ∎

5. The Stationary Problem with ϕ' Bounded

Theorem 2. *Let $\phi \in C'(\mathbb{R}, \mathbb{R}^m)$ with ϕ' bounded and uniformly continuous, and let λ, $\epsilon > 0$. Then for each $h \in X$ with $h\,|_{Q_p} \in L^2(Q_p)$, there is a $u \in X$ such that $u\,|_{Q_p} \in H^2(Q_p)$ and*

$$u + \lambda \phi(u)_x - \epsilon \Delta u = h. \tag{11}$$

Proof. Define B by $Bu = \phi(u)_x = \phi'(u) \cdot u_x$ for u in $X_1 = \{u \in X : u\,|_{Q_p} \in H^1(Q_p)\}$; X_1 is a Banach space under the norm $\| u \|_{H^1} = \| u \|_{H^1(Q_p)}$. For future use we define $X_2 = \{u \in X : u\,|_{Q_p} \in H^2(Q_p)\}$, which is a Banach space under the norm $\| u \|_{H^2} = \| u \|_{H^2(Q_p)}$.

View B as a mapping from X_1 to X^2, where $X^2 = \{u \in X : u\,|_{Q_p} \in L^2(Q_p)\}$ is a Banach space under the norm $\| u \|_2 = \| u \|_{L^2(Q_p)}$. Since ϕ' is bounded we have

$$\| \phi_i'(u)\, u_{x_i} \|_2^2 \leqslant k_0 \| u_{x_i} \|_2^2,$$

for some constant k_0. Hence we have the estimate

$$\| Bu \|_2 \leqslant m k_0 \| u \|_{H^1}. \tag{12}$$

We will use the following lemma to get the existence of the solution of (11).

LEMMA 4. *Let H be a Hilbert space, T_1 a non-negative self-adjoint linear operator in H and T_2: $D(T_2) \subseteq H \to H$ be a single-valued operator with properties $D(T_2) \supseteq D(T_1)$ and satisfy the following four conditions:*

(i) $(T_2(u), u) = 0$ *for* $u \in D(T_1)$.

(ii) *There is a* $k < 1$ *and a function* K: $\mathbb{R}^+ \to \mathbb{R}$ *such that* $u \in D(T_1)$ *and* $\| u \| \leqslant M$ *imply* $\|(I + T_2)(u)\| \leqslant k \| T_1(u)\| + K(M)$, *where* I: $H \to H$ *is the identity map.*

(iii) *If* $\Gamma \subseteq D(T_1)$ *and* $T_1\Gamma$ *are bounded, then* T_2: $\Gamma \to Y$ *is continuous from the relative weak topology on* Γ *to the weak topology on* H.

(iv) $$\lim_{\substack{u \in D(T_1) \\ \|u\| \to \infty}} \frac{(u + T_1 u + T_2 u, u)}{\| u \|} = \infty.$$

Then the map $u \mapsto u + T_2(u) + T_1(u)$, $u \in D(T_1)$ *is onto* H.

Lemma 4 is a very special case of Theorem 2 of *H.* Brezis' paper [2]. We will not prove it here.

Returning to the proof of Theorem 2, let $H = X^2$, $T_2 = \lambda B$, $T_1 = -\epsilon \Delta$ on $D(T_1) = \{u \in X: u|_{Q_p} \in H^2(Q_p)\}$; here Δ has periodic boundary conditions, so T_1 is nonnegative self-adjoint.

In order to apply the previous lemma, we prove that the conditions (i), (ii), (iii), (iv) are satisfied. First we show that $(T_2 u, u) = \lambda(Bu, u) = 0$. Since

$$\int_0^{p_i} \phi_i'(u)\, u u_{x_i}\, dx_i = \int_0^{p_i} \left(\int_0^u \phi_i'(s)\, s\, ds\right)_{x_i} dx_i = \int_{r_i}^{p_i + r_i} \left(\int_0^u \phi_i'(s)\, s\, ds\right)_{x_i} dx_i$$

$$= \int_0^{u(\overline{p_i + r_i})} \phi_i'(s)\, s\, ds - \int_0^{u(\bar{r}_i)} \phi_i'(s)\, s\, ds = 0$$

by periodicity, for a.e. $(\xi_1, \ldots, \xi_{i-1}, \xi_{i+1}, \ldots, \xi_m)$ where $\bar{r}_i = (\xi_1, \ldots, \xi_{i-1}, r_i, \xi_{i+1}, \ldots, \xi_m)$ and $\overline{p_i + r_i} = (\xi_1, \ldots, \xi_{i-1}, p_i + r_i, \xi_{i+1}, \ldots, \xi_m)$ are points of continuity of u. It follows that

$$(Bu, u) = \int_{Q_p} \phi'(u) \cdot u u_x\, dx = \sum_{i=1}^m \int_{Q_p} \phi_i'(u)\, u u_{x_i}\, dx = 0.$$

Next we prove (ii). Since we have (see [9]), for some fixed $C > 0$ and sufficiently small $\gamma > 0$,

$$\int_{Q_p} | u_{x_i} |^2\, dx \leqslant \gamma \int_{Q_p} | u_{x_i x_i} |^2\, dx + \frac{C}{\gamma} \int_{Q_p} | u |^2\, dx$$

for $u \in X$ such that $u|_{Q_p} \in H^2(Q_p)$, hence we get

$$\| u_{x_i} \|_2 \leqslant \gamma \| u_{x_i x_i} \|_2 + \frac{C}{\gamma} \| u \|_2$$

for all $\gamma > 0$ and some (possibly different) $C > 0$. It follows that

$$\sum_{i=1}^{m} \| u_{x_i} \|_2 \leqslant \gamma \sum_{i=1}^{m} \| u_{x_i x_i} \|_2 + \frac{mC}{\gamma} \| u \|_2 \leqslant \gamma \| u \|_{H^2} + \frac{mC}{\gamma} \| u \|_2 .$$

But there is a constant C_1 such that for all $u \in X$ with $u|_{Q_p} \in H^2(Q_p)$, we have

$$\| u \|_{H^2} \leqslant C_1(\| \Delta u \|_2 + \| u \|_2).$$

(See [3].) Hence we get

$$\sum_{i=1}^{m} \| u_{x_i} \|_2 \leqslant \gamma C_1 \| \Delta u \|_2 + \left(\gamma C_1 + \frac{mC}{\gamma}\right) \| u \|_2 .$$

It follows that

$$\| u + \lambda\phi(u)_x \|_2 \leqslant \lambda C_1 C_2 \gamma \| \Delta u \|_2 + C_3 \left[\lambda C_2 \left(\gamma C_1 + \frac{mC}{\gamma}\right) + 1\right] \| u \|_2 ,$$

where $C_2 = \max_{r \in \mathbb{R}, i=1,\ldots,m} | \phi_i(r)|$ and C_3 is a fixed constant satisfying $\| u \|_2 \leqslant C_3 \| u \|_2$ for all $u \in X$ with $u|_Q \in L^2(Q_p)$. We choose γ small enough such that $k = \lambda C_1 C_2 \gamma < 1$. This verifies (ii).

For (iii), let $\{u_l\}$ and $\{-\epsilon \Delta u_l\}$ are bounded in X^2, thus $\{u_l\}$ is bounded in $X_2{}^2$. Then by an imbedding theorem (see [7]), we can assume $u_l \to u$ in $X_1{}^2$; otherwise replace $\{u_l\}$ by a convergent subsequence. Since $\{Bu_l\}$ is bounded by (13), the proof of (iii) will be complete if we show $Bu_l \to v$ weakly in X^2 implies $v = Bu$ weakly in X^2. To see this, we need another lemma.

LEMMA 5. *Let $u_l \to u$ in X^2 as $l \to \infty$ and let ϕ' be bounded and uniformly continuous. Then $(\phi_i(u_l))_{x_i} \to (\phi_i(u))_{x_i}$ weakly in X^2 as $l \to \infty$ for each $i = 1,\ldots, m$.*

Proof. Let $v \in X^2$. Since $f_n \to f$ in X^2 as $n \to \infty$ if and only if $\int_{Q_p} g_n \to 0$ as $n \to \infty$ where $g_n = |f_n - f|^2 \geqslant 0$, these equivalent conditions imply that for any positive integer j, there is an $N(j) > 0$ such that $0 \leqslant \int_{Q_p} g_k < 1/j^2$ for $k \geqslant N(j)$. Let $B_{jk} = \{x \in Q_p \mid g_k(x) \geqslant 1/j\}$. Then we have $0 \leqslant (1/j)\,\mu(B_{jk}) \leqslant \int_{B_{jk}} g_k \leqslant \int_{Q_p} g_k < 1/j^2$, hence $\mu(B_{jk}) < 1/j$ for $k \geqslant N(j)$. And so as $j \to \infty$, $\int_{B_{jk}} |v|^2\,dx \to 0$ for $v \in L^2(Q_p)$. Take $f_k = u_k$, $f = u$ above. Thus for any j, $k \geqslant N(j)$ implies $|u_k(x) - u(x)|^2 < 1/j$ for $x \in Q_p - B_{jk}$, since $u_k \to u$ in X^2 as $k \to \infty$.

Let $v \in L^2(Q_p)$ be fixed but arbitrary. Since ϕ_i' is bounded, for any n, there is a j_0 such that $\int_{B_{jk}} |v|^2\,dx < 1/n(2M + \| v \|_2^2)$, where $M = \sup_{x,i} |\phi_i'(x)|^2$, for $j \geqslant j_0$. Moreover we may suppose $|x - y|^2 < 1/j_0$ implies $|\phi_i(x) - \phi_i(y)|^2 < 1/n(2M + \| v \|_2^2)$ since ϕ_i' is uniformly continuous on $\mathbb{R}$. Now, for any n, take $j = j_0$ and $N(j)$ as above. Then for all $k \geqslant N(j)$, we have for $I_k \equiv |(\phi_i'(u_k) - \phi_i'(u))\, v|^2$,

$$\begin{aligned}
&\int_{Q_p} |(\phi_i'(u_k) - \phi_i'(u))\, v|^2\, dx \\
&\quad\leqslant \int_{B_{jk}} I_k\, dx + \int_{Q_p \backslash B_{jk}} I_k\, dx \\
&\quad\leqslant 2M \int_{B_{jk}} |v|^2\, dx + \frac{1}{n(2M + \| v \|_2^2)} \int_{Q_p \backslash B_{jk}} |v|^2\, dx < \frac{1}{n}
\end{aligned}$$

by our choice of j. Hence

$$|(u_{x_i}, (\phi_i'(u_k) - \phi_i'(u))\, v)| \leqslant \| u_{x_i} \|_2 \|(\phi_i'(u_k) - \phi_i'(u))\, v\|_2 .$$

Note that $(\phi_i'(u_k) - \phi_i'(u))\, v \in X^2$. Thus

$$\begin{aligned}
&|(\phi_i'(u_k)\, u_{k_{x_i}} - \phi_i'(u)\, u_{x_i}, v)| \\
&\quad\leqslant \| \phi_i'(u_k)\,(u_{k_{x_i}} - u_{x_i})\|_2 \| v \|_2 + |(u_{x_i}, (\phi_i'(u_k) - \phi_i'(u))\, v)| \\
&\quad\leqslant M \| u_{k_{x_i}} - u_{x_i} \|_2 \| v \|_2 + \| u_{x_i} \|_2 \|(\phi_i'(u_k) - \phi_i'(u))\, v \|_2
\end{aligned}$$

for $v \in X^2$, and the right hand side approaches zero as $k \to \infty$. Hence Lemma 5 is proved. ∎

We return to the application of Lemma 4. By

$$|(\phi'(u_k) \cdot u_{k_x} - \phi'(u) \cdot u_x, v)| \leqslant \sum_{i=1}^{m} |(\phi_i'(u_k)\, u_{k_{x_i}} - \phi_i'(u)\, u_{x_i}, v)|$$

we see that $\phi'(u_k) \cdot u_{k_x} \to \phi'(u) \cdot u_x$ weakly and that $Bu_l \to v$ weakly implies $v = Bu$ weakly. Hence (iii) is verified.

For (iv), we have

$$-\int_0^{p_i} u_{x_i x_i} u\, dx_i = \int_0^{p_i} u_{x_i}^2\, dx_i$$

by periodicity as before. Thus we have $(-\epsilon \Delta u, u) = -\epsilon \sum_{i=1}^{m} \int_{Q_p} u_{x_i x_i} u\, dx = \epsilon \| u_x \|_2^2$. Since $(T_1 u, u) = (-\epsilon \Delta u, u) = \epsilon \| u_x \|_2^2 \geqslant 0$ and $(T_2 u, u) = 0$, we have

$$\frac{(u + T_1 u + T_2 u, u)}{\| u \|_2} \geqslant \frac{C_1 \| u \|_{H^1}^2}{\| u \|_2} \geqslant C_2 \| u \|_2 \to \infty$$

as $\| u \|_2 \to \infty$, where C_1, C_2 are constants independent of u, and for the second inequality see [7]. Thus (iv) is verified.

Hence the hypothesis of the Lemma 4 is verified, and since Theorem 2 follows immediately from Lemma 4, the proof of the theorem is complete. ∎

LEMMA 6. *Let the assumptions of Theorem 2 hold. Let $u \in X$ with $u|_{Q_p} \in H^2(Q_p)$ satisfy* (11), *where $h \in X$ is such that $h|_{Q_p} \in L^q(Q_p) \cap L^2(Q_p)$ and $\| u \|_q \leqslant \| h \|_q$.*

Proof. We treat the case $1 \leqslant q < \infty$ first. Choose a sequence of monotone, piecewise smooth, odd functions $\{\alpha_{q,l}\}_{l=1}^{\infty}$ such that

$$\begin{aligned} &\text{(i)} \quad \lim_{l\to\infty} \alpha_{q,l}(r) = | r |^{q-1} \operatorname{sign}_0 r \qquad \text{for } r \in \mathbb{R}, \\ &\text{(ii)} \quad | r |^{q-1} \geqslant | \alpha_{q,l}(r)| \qquad \text{for } r \in \mathbb{R}. \end{aligned} \tag{13}$$

The property (13) may be restated as

$$r\alpha_{q,l}(r) \geqslant | \alpha_{q,l}(r)|^{q'} \qquad q' = \frac{q}{q-1}, \quad \text{if} \quad q > 1. \tag{14}$$

Multiplying (11) by $\alpha_{q,l}(u)$ and integrating over Q_p gives us

$$\begin{aligned} &\int_{Q_p} [u\alpha_{q,l}(u) + \lambda\phi(u)_x\, \alpha_{q,l}(u) - \epsilon(\Delta u)\, \alpha_{q,l}(u)]\, dx \\ &\qquad = \int_{Q_p} h\alpha_{q,l}(u)\, dx \leqslant \| h \|_q \| \alpha_{q,l}(u)\|_{q'}, \end{aligned} \tag{15}$$

where the right hand side is finite by (13) or (14), $u|_{Q_p}$ and $\alpha_{q,l}(u)|_{Q_p} \in L^2(Q_p)$ for the following reasons. First, since (13) (ii) implies $| r |^q = | r |^{(q-1)q'} \geqslant | \alpha_{q,l}(r)|^{q'}$, thus if $q = 2$, $q' = 2$, we have $u|_{Q_p} \in L^2(Q_p)$, which implies $\alpha_{q,l}(u)|_{Q_p} \in L^2(Q_p)$. If $q \neq 2$, by (11), we have $u\alpha_{q,l}(u) \geqslant | \alpha_{q,l}(u)|^{q'}$ or

$$\int_{Q_p} | \alpha_{q,l}(u)|^{q'}\, dx \leqslant \int_{Q_p} u\alpha_{q,l}(u)\, dx \leqslant \| u \|_2 \| \alpha_{q,l}(u)\|_2 < \infty \tag{16}$$

by (13) (ii). Also

$$\int_{Q_p} (\Delta u)\, \alpha_{q,l}(u)\, dx = -\int_{Q_p} \alpha'_{q,l}(u)\, | u_x |^2\, dx \leqslant 0 \tag{17}$$

since $\alpha_{q,l}$ is monotone, where in the equality we have used the fact

$$\begin{aligned} &\int_0^{p_i} u_{x_i x_i} \alpha_{q,l}(u)\, dx_i + \int_0^{p_i} | u_{x_i} |^2\, \alpha'_{q,l}(u)\, dx_i \\ &\qquad = \int_0^{p_i} [u_{x_i} \alpha_{q,l}(u)]_{x_i}\, dx_i = \int_{r_i}^{p_i + r_i} [u_{x_i} \alpha_{q,l}(u)]_{x_i}\, dx_i = 0 \end{aligned}$$

here again $p_i + r_i$, r_i are points of continuity of u and u_{x_i}, and we have used the periodicity of u. This gives

$$\int_{Q_p} (\Delta u)\, \alpha_{q,l}(u)\, dx = -\int_{Q_p} \alpha'_{q,l}(u) \sum_{i=1}^{m} u_{x_i}^2\, dx = -\int_{Q_p} \alpha'_{q,l}(u)\, |\, u_x\, |^2\, dx.$$

Also,

$$\begin{aligned} \int_0^{p_i} \phi_i'(u)\, u_{x_i} \alpha_{q,l}(u)\, dx_i &= \int_0^{p_i} \left(\int_0^u \alpha_{q,l}(s)\, \phi_i'(s)\, ds\right)_{x_i} dx_i \\ &= \int_{r_i}^{p_i+r_i} \left(\int_0^u \alpha_{q,l}(s)\, \phi_i'(s)\, ds\right)_{x_i} dx_i = 0 \end{aligned} \tag{18}$$

consequently we have $\int_{Q_p} \phi(u)_x\, \alpha_{q,l}(u)\, dx = 0$. Using (16), (17), (18) in (15), we get $\int_{Q_p} |\, \alpha_{q,l}(u)|^{q'}\, dx \leqslant \|\, h\, \|_q \,\|\, \alpha_{q,l}(u)\|_{q'}$ or $(\int_{Q_p} |\, \alpha_{q,l}(u)|^{q'}\, dx)^{1/q} \leqslant \|\, h\, \|_q$. Since $|\, \alpha_{q,l}(u)|^{q'} \to u^q$ as $l \to \infty$, the result follows from Fatou's lemma.

Next we treat the case $q = \infty$. If $M \geqslant h^+$ a.e., then subtract M from both sides of (11), and multiplication by $(u - M)^+$ (the positive part of $u - M$) yields

$$\begin{aligned} (u - M)(u - M)^+ + \lambda\phi(u)_x (u - M)^+ - \epsilon(\Delta u)(u - M)^+ \\ = (h - M)(u - M)^+ \leqslant 0. \end{aligned}$$

Since

$$\begin{aligned} \int_0^{p_i} \phi_i'(u)\, u_{x_i}(u - M)^+\, dx_i &= \int_0^{p_i} \left(\int_0^u \phi_i'(s)\,(s - M)^+\, ds\right)_{x_i} dx_i \\ &= \int_{r_i}^{p_i+r_i} \left(\int_0^u \phi_i'(s)\,(s - M)^+\, ds\right)_{x_i} dx_i = 0 \end{aligned}$$

we get $\int_{Q_p} \phi(u)_x (u - M)^+\, dx = 0$. By

$$\begin{aligned} \int^{p_i} u_{x_i x_i}(u - M)^+\, dx_i &= \int_{r_i}^{p_i+r_i} u_{x_i x_i}(u - M)^+\, dx_i \\ &= [u_{x_i}(u - M)^+]_{r_i}^{\overline{p_i+r_i}} - \int_{r_i}^{p_i+r_i} u_{x_i}(u - M)^+_{x_i}\, dx_i \\ &= -\int_{r_i}^{p_i+r_i} u_{x_i}^2 \frac{d}{ds}(s - M)^+ \Big|_{s=u} dx \leqslant 0, \end{aligned}$$

since

$$\frac{d}{ds}(s - M)^+(s) = \begin{cases} 1 & \text{if} \quad s > M \\ 0 & \text{if} \quad s < M \end{cases}$$

and since $u_{x_i} = 0$ a.e. on $\{x: u(x) = M\}$. (See [18].) Thus $\int_{Q_p} u_{x_i x_i}(u - M)^+ \, dx \leqslant 0$ or $\int_{Q_p} (\Delta u)(u - M)^+ \, dx \leqslant 0$. So

$$\int_{Q_p} [(u - M)^+]^2 \, dx = \int_{Q_p} (u - M)^+ (u - M)^- \, dx + \int_{Q_p} [(u - M)^+]^2 \, dx$$

$$= \int_{Q_p} (u - M)(u - M)^+ \, dx \leqslant 0$$

which implies $(u - M)^+ \leqslant 0$ or $u \leqslant M$ a.e.

A similar estimate gives us that if $M \geqslant h^- = \max(0, -h)$, then $-M \leqslant u$ a.e. Since we can take $M = \| h \|_\infty$, the proof is complete. ∎

For m satisfying $1 \leqslant m \leqslant 5$ and $h \in C_p^\infty$, if u satisfies (11), then $u \in C(\mathbb{R}^m)$ (after correction on a small set), by an argument using the Sobolov imbedding theorem; we omit the proof. For $m \geqslant 6$ we require more machinery to establish continuity of u. We shall use the following theorem from [14].

THEOREM 3. *Assume that the function $a_i(x, u, r)$, $i = 1, \ldots, m$, and $a(x, u, r)$ are defined for $x \in \bar{\Omega} \subset \mathbb{R}^m$, $u \in \mathbb{R}$, $r \in \mathbb{R}^m$, and that they satisfy the conditions:*

$$\sum_{i=1}^{m} | a_i(x, u, r)| (1 + | r |) + | a(x, u, r)| \leqslant \mu(| u |)(1 + | r |)^n, \tag{*}$$

$$\sum_{i=1}^{m} a_i(x, u, r) r_i \leqslant \nu(| u |) | r |^n - \mu(| u |) \tag{**}$$

where $| r | = (\sum_{i=1}^m r_i^2)^{1/2}$ and $\mu(t)$ is a positive nondecreasing continuous function, and $\nu(t)$ is a positive nonincreasing continuous function, and the constant n satisfies $n > 1$. Here Ω is a bounded region with a smooth boundary. Then an arbitrary bounded generalized solution $u(x)$ of

$$\sum_{i=1}^{m} \frac{\partial}{\partial x_i} a_i(x, u, u_x) + a(x, u, u_x) = 0$$

belongs (after correction on a set of measure zero) to the class $C_{0,\alpha}(\Omega)$ of Hölder continuous functions with exponent $\alpha > 0$ depending on $M = \operatorname{ess\,max}_\Omega | u(x)|$ and the constants n, $\nu(M)$, and $\mu(M)$.

COROLLARY 1. *Let the hypothesis of Theorem 2 be satisfied and $h |_{Q_p} \in L^\infty(Q_p)$. Then if u satisfies (11), it follows that $u \in C(Q_p)$ after correction on a null set.*

Proof. Choose an open ball B_ρ in $\mathbb{R}^m$ centered at the origin and with radius ρ such that $\bar{Q}_p \subset B_\rho$. Letting $a_i(x, u, u_x) = \epsilon u_{x_i}$, $a(x, u, u_x) = h - u - \lambda \sum_{i=1}^m \phi_i'(u) u_{x_i}$ the corollary follows easily from the above theorem and Lemma 6.

THEOREM 4. *Let the assumptions of Theorem 2 hold and let* $u, v \in X$ *with* $u|_{Q_p}, v|_{Q_p} \in H^2(Q_p) \cap C(Q_p)$ *satisfy*

$$u + \lambda\phi(u)_x - \epsilon\Delta u = h$$
$$v + \lambda\phi(v)_x - \epsilon\Delta v = g$$

for $\epsilon > 0$. *If* $g, h \in X$ *with* $g|_{Q_p}, h|_{Q_p} \in L^1(Q_p)$, *then*

$$\|(u - v)^+\|_1 \leqslant \|(h - g)^+\|_1 .$$

Proof. Define

$$\Phi_l(s) = \begin{cases} \dfrac{l}{2} s^2 + \dfrac{1}{2l} & \text{if} \quad |s| \leqslant \dfrac{1}{l} \\ |s| & \text{if} \quad |s| \geqslant \dfrac{1}{l}, \end{cases}$$

and define ψ_l by $\psi_l' = (\Phi_l')^+$, $\psi_l(0) = 0$. Let $1 \geqslant f \geqslant 0$, $f \in C_0^\infty[Q_p]$ and $w = u - v$. Then

$$w + \lambda(\phi(u) - \phi(v))_x - \epsilon\Delta w = h - g.$$

Multiplying the above by $\psi_l'(w) f$ and integrating over Q_p yields

$$\begin{aligned} \int_{Q_p} \{w\psi_l'(w) f + \lambda(\phi(u) - \phi(v))_x \psi_l'(w) f - \epsilon(\Delta w)\psi_l'(w) f\}\, dx \\ \leqslant \|(h - g)^+\|_1 , \end{aligned} \tag{19}$$

since $0 \leqslant \psi_l' f \leqslant 1$ and $\int_{Q_p} \psi_l' f(h - g)\, dx \leqslant \|(h - g)^+\|_1$. Now $\psi_l'(w) \in X_1^2$, $\psi_l'' \geqslant 0$, $f \geqslant 0$, so by $[w_{x_i}\psi_l'(w) f]_{x_i=0}^{x_i=p_i} = 0$, for $f \in C_0^\infty[Q_p]$, we have

$$\begin{aligned} \int_0^{p_i} w_{x_i x_i}\psi_l'(w) f\, dx_i &= -\int_0^{p_i} \psi_l''(w)\, w_{x_i}^2 f\, dx_i + \int_0^{p_i} \psi_l(w) f_{x_i x_i}\, dx_i \\ &\leqslant \int_0^{p_i} \psi_l(w) f_{x_i x_i}\, dx_i , \end{aligned}$$

thus we get $\int_{Q_p} (\Delta w)\, \psi_l'(w) f\, dx \leqslant \int_{Q_p} \psi_l(w)\, \Delta f\, dx$.

Letting $l \to \infty$ yields

$$\limsup_{l\to\infty} \int_{Q_p} -(\Delta w)\, \psi_l'(w) f\, dx \geqslant -\int_{Q_p} w^+ \Delta f\, dx \tag{20}$$

where we have used the fact $w|_{Q_p} \in L^1(Q_p)$ by Lemma 6. Next

$$\int_{Q_p} (\phi(u) - \phi(v))_x \psi_l'(w) f \, dx$$

$$= - \int_{\Omega_l} \psi_l''(w) (\phi(u) - \phi(v)) \cdot w_x f \, dx - \int_{Q_p} \psi_l'(w) (\phi(u) - \phi(v)) \cdot f_x \, dx \tag{21}$$

where $\Omega_l = \{x \in Q_p : |u(x) - v(x)| \leqslant 1/l\}$ and we use the fact that $\psi_l''(w) = 0$ off Ω_l and the boundary terms vanish by $f \in C_0^\infty[Q_p]$. For the first term we note that

$$|\psi_l''(w) (\phi(u) - \phi(v))| \leqslant lK |u - v| \leqslant K$$

on Ω_l, where $K \geqslant |\phi'|$ is fixed. Hence by the dominated convergence theorem,

$$\limsup_{l \to \infty} \left| \int_{\Omega_l} \psi_l''(w) (\phi(u) - \phi(v)) \cdot w_x f \, dx \right| \leqslant K \int_{\Omega} |w_x| f \, dx \tag{22}$$

where $\Omega = \bigcap_l \Omega_l$. But $u = v$ on Ω implies $w_x = 0$ a.e. on Ω, so the integral on the right is zero. Thus by (22) and taking a suitable subsequence and applying Lemma 2 to (21), we get

$$\limsup_{l \to \infty} \int_{Q_p} \psi_l'(w) (\phi(u) - \phi(v))_x f \, dx \geqslant - \int_{Q_p} \operatorname{sgn}(w^+) (\phi(u) - \phi(v)) \cdot f_x \, dx. \tag{23}$$

Using (20), (23) in (19) and letting $l \to \infty$ yields

$$\begin{aligned} \int_{Q_p} w^+ f \, dx - \epsilon \int_{Q_p} w^+ (\Delta f) \, dx - \int_{Q_p} \operatorname{sgn}(w^+) (\phi(u) - \phi(v)) \cdot f_x \, dx \\ \leqslant \|(g - h)^+\|_1, \end{aligned} \tag{24}$$

for all $f \in C_0^\infty[Q_p]$, $1 \geqslant f \geqslant 0$.

We need the following two lemmas. Their proof are similar to the proof of Lemma 3, we omit it. (See [16].)

LEMMA 7. *For any $\epsilon > 0$, there is an $f \in C_0^\infty[Q_p]$ such that*

$$\left| \int_{Q_p} (\operatorname{sign}_0 w^+) (\phi(u) - \phi(v)) \cdot f_x \, dx \right| < \epsilon$$

for each fixed $u, v \in C(Q_p)$, where $w = u - v$ and w^+ is the positive part of $u - v$.

LEMMA 8. *In Lemma 7, we can pick $f \in C_0^\infty[Q_p]$ with $f(\xi) = \prod_{i=1}^m f_i(\xi_i)$ satisfying the following:*

$$\frac{\partial^2 f_i}{\partial \xi_i^2}(x_0) = \frac{\partial^2 f_i}{\partial \xi_i^2}(p_i - x_0) = 0 \qquad \text{and} \qquad \frac{\partial^2 f_i}{\partial \xi_i^2}\bigg|_{I_i^1} \geqslant 0,$$

$$\frac{\partial^2 f_i}{\partial \xi_i^2}\bigg|_{I_i^2} \leqslant 0, \qquad \frac{\partial^2 f_i}{\partial \xi_i^2}\bigg|_{I_i^3} \geqslant 0, \qquad \frac{\partial^2 f_i}{\partial \xi_i^2}\bigg|_{I_i^4} \leqslant 0$$

are symmetric with respect to $x_0 - \delta/2$, $x_0 + \delta/2$, $p_i - x_0 - \delta/2$, $p_i - x_0 + \delta/2$ respectively, where $I_i^1 = [x_0 - \delta, x_0]$, $I_i^2 = [x_0, x_0 + \delta]$, $I_i^3 = [p_i - x_0 - \delta, p_i - x_0]$, $I_i^4 = [p_i - x_0, p_i - x_0 + \delta]$. Moreover, f can also be chosen to satisfy

$$\left| \int_{Q_p} w^+ \Delta f \, dx \right| < \epsilon$$

for any given $\epsilon > 0$.

Now choose a sequence $\{\epsilon_k\}$ of positive numbers decreasing to zero and the corresponding $\{f^k\} \subset C_0^\infty[Q_p]$ satisfying Lemmas 7 and 8. Set $f = f^k$ in (24) and let $k \to \infty$ to obtain

$$\int_{Q_p} w^+ \, dx = \int_{Q_p} (u - v)^+ \, dx \leqslant \|(g - h)^+\|_1 .$$

The proof of Theorem 4 is complete. ∎

COROLLARY 2. *Let the assumptions of Theorem 2 be satisfied. Let $u, v \in X$ be such that $u|_{Q_p}, v|_{Q_p} \in H^2(Q_p) \cap C(Q_p)$ and*

$$u + \lambda(\phi(u))_x - \epsilon \Delta u = h$$

$$v + \lambda(\phi(v))_x - \epsilon \Delta v = g$$

where $h, g \in X$. Then $\| u - v \|_1 \leqslant \| u - v \|_1$. Moreover, if $g \geqslant h$ a.e., then $v \geqslant u$ a.e.

Proof. Take $\psi_l' = (\Phi_l')^-$ in the proof of the above theorem; the arguments in the proof of Theorem 4 yield $\|(u - v)^-\|_1 \leqslant \|(h - g)^-\|_1$. Then $g \geqslant h$ a.e. implies $v \geqslant u$ a.e. Moreover,

$$\| u - v \|_1 = \|(u - v)^+\|_1 + \|(u - v)^-\|_1 \leqslant \|(h - g)^+\|_1 + \|(h - g)^-\|$$
$$= \| h - g \|_1 . \quad ∎$$

6. The Range Condition

Let $X_0 = \{u \in X : u|_{Q_p} \in C(Q_p)\} = X \cap C(\mathbb{R}^m)$.

Theorem 5. *Let $\phi \in C^1$. Let A_0 be given by Definition 1. Then $R(I - \lambda A_0) \supseteq X_0$ for $\lambda > 0$. Let $T_\lambda : X_0 \to X^1$ be the restriction of $(I - \lambda A_0)^{-1}$ to X_0. If $g, h \in X_0$, then the following statements hold:*

(i) *For $1 \leqslant q \leqslant \infty$, $h \in X^q$, $T_\lambda h \in X^q$ and $\| T_\lambda h \|_q \leqslant \| h \|_q$.*

(ii) $-\| h^- \|_\infty \leqslant T_\lambda h \leqslant \| h^+ \|_\infty$ *a.e.*

(iii) $\|(T_\lambda h - T_\lambda g)^+\|_1 \leqslant \|(h - g)^+\|_1$.

Proof. We may assume without loss of generality that $\phi(0) = 0$. Choose a sequence $\{\phi^j\}_{j=1}^\infty$ of C^1 functions from $\mathbb{R}$ to $\mathbb{R}^m$ such that $\phi^j(0) = 0$, $(\phi^j)'$ is bounded on $\mathbb{R}$, and $\{\phi^j\}_{j=1}^\infty$ converges to ϕ in $W^{1,\infty}_{\mathrm{loc}}(\mathbb{R})$, that is, $\{\phi^j\}_{j=1}^\infty$ converges to ϕ and $(\phi^j)'$ converges to ϕ' uniformly on compact sets. Given $\lambda > 0$ and $j = 1, 2, \ldots$ define $T_{\lambda,j} : X_0 \to X$ by $T_{\lambda,j} h = u$ and $u + \lambda \phi^j(u)_x - (1/j)\,\Delta u = h$.

$T_{\lambda,j}$ is well-defined by Theorems 2, 4 and Lemma 6 and it has the properties claimed for T_λ in this theorem by Lemma 6, Theorem 4, and Corollary 2. By Corollary 2 and the translation invariance of $T_{\lambda,j}$, we have $\int_{Q_p} | u_j(x + y) - u_j(x)|\, dx \leqslant \int_{Q_p} | h(x + y) - h(x)|\, dx$ holds for all $y \in \mathbb{R}^m$. Hence $\{u_j\}$ is relatively compact in X^q for $1 \leqslant q < \infty$. (See [8].) Thus there is a subsequence $\{u_{j(i)}\}$ of $\{u_j\}$ which converges a.e. and in X to a limit $u \in X$. We denote this convergence by $\twoheadrightarrow$; thus $u_{j(i)} \twoheadrightarrow u$. Let $f \in C_0^\infty[Q_p]$ and $\Phi = \Phi_l$ for some l, where the function Φ_l is defined as in the proof of Theorem 4. Multiply by $\Phi'(u_j) f$ the equation satisfied by u_j. Integrating over Q_p yields

$$\int_{Q_p} \Big\{ u_j \Phi'(u_j) f - \lambda(\Phi'(u_j)\,\phi^j(u_j) - \Phi'(k)\,\phi^j(k)) \cdot f_x + \lambda \left(\int_k^{u_j} \Phi''(s)\,\phi^j(s)\, ds \right) \cdot f_x$$

$$+ \frac{1}{j} (\Phi''(u_j)\,|\, u_{jx}\,|^2 f - \Phi(u_j)\,\Delta f) \Big\}\, dx$$

$$= \int_{Q_p} h \Phi'(u_j)\, f\, dx$$

for all $k \in \mathbb{R}$, where we have used

$$\Phi'(u_j)\,\phi^j(u_j)_x = \left(\int_k^{u_j} \Phi'(s)\,(\phi^j)'(s)\, ds \right)_x$$

$$= \left([\Phi'(s)\,\phi^j(s)]_{s=k}^{s=u_j} - \int_k^{u_j} \Phi''(s)\,\phi^j(s)\, ds \right)_x$$

$$= \left(\Phi'(u_j)\,\phi^j(u_j) - \Phi'(k)\,\phi^j(k) - \int_k^{u_j} \Phi''(s)\,\phi^j(s)\, ds \right)_x$$

and

$$\int_{Q_p} (\Delta u_j)\,\Phi'(u_j) f\,dx = -\int_{Q_p} |\,u_{jx}\,|^2\,\Phi''(u_j) f\,dx - \int_{Q_p} u_{jx}\cdot \Phi'(u_j) f_x\,dx$$

$$= -\int_{Q_p} |\,u_{jx}\,|^2\,\Phi''(u_j) f\,dx + \int_{Q_p} \Phi(u_j)\,\Delta f\,dx,$$

where we have used

$$\int_{Q_p} u_{jx}\cdot \Phi'(u_j) f_x\,dx = \int_{Q_p} \Phi(u_j)_x\cdot f_x\,dx = -\int_{Q_p} \Phi(u_j)\,\Delta f\,dx.$$

Since $\Phi'' \geqslant 0$ and $f \geqslant 0$, the term involving $\Phi''(u_j)\,|\,u_{jx}\,|^2 f$ is nonnegative. Moreover $\|\,u_j\,\|_\infty \leqslant \|\,h\,\|_\infty$, so $\int_{Q_p} \Phi(u_j)\,\Delta f\,dx$ is bounded in j.

Letting j tend to ∞ through the subsequence $\{j(i)\}$ and using convergence $u_{j(i)} \twoheadrightarrow u$ and $\phi_j \to \phi$, $(\phi^j)' \to \phi'$ uniformly on compact sets yields

$$\int_{Q_p} \Big\{u\Phi'(u) f - \lambda(\Phi'(u)\,\phi(u) - \Phi'(k)\,\phi(k))\cdot f_x + \lambda\left(\int_k^u \Phi''(s)\,\phi(s)\,ds\right)\cdot f_x\Big\}\,dx$$

$$\leqslant \int_{Q_p} h\Phi'(u) f\,dx$$

for $f \in C_0^\infty[Q_p]$, $f \geqslant 0$, $k \in \mathbb{R}$. Next let $\Phi(s) = \Phi_l(s-k)$ and $l \to \infty$. Since

$$\int_k^u \Phi_l''(s-k)\,\phi(s)\,ds = \operatorname{sign}_0(u-k)\,\phi(k)$$

hence

$$\int_{Q_p} \operatorname{sign}_0(u-k)\,\{uf - \lambda(\phi(u) - \phi(k))\cdot f_x - hf\}\,dx \leqslant 0$$

for all $k \in \mathbb{R}$ and nonnegative f in $C_0^\infty[Q_p]$. Since u is unique by dissipativeness of A_0, it follows that $\lim_{j\to\infty} T_{\lambda,j}h \equiv T_\lambda h$ holds with convergence in X. The conditions that have been established for $T_{\lambda,j}$ are preserved under L^1 convergence, and so the proof is complete. ∎

MAIN THEOREM. *Let $\phi \in C^1(\mathbb{R}, \mathbb{R}^m)$. Then the closure A of A_0 satisfies the assumption of the Crandall–Liggett generation theorem* [5].

Proof. First we note that the closure A of A_0 is dissipative since A_0 is. Moreover, if $h \in X$, $h\,|_{Q_p} \in L^1(Q_p)$, let $\{h_k\} \subset X_0$ satisfy $h_k \to h$; this can be done since X_0 is dense in X. Then $\{T_\lambda h_k\}$ is Cauchy in X since T_λ is a contraction and $T_\lambda 0 = 0$.

Let $w_k = (h_k - T_\lambda h_k)/\lambda$. If $u_k = T_\lambda h_k$, then $(I - \lambda A_0)\,u_k = h_k$, hence $u_k - \lambda A_0 u_k = h_k$ or $(h_k - u_k)/\lambda = -A_0 u_k = -A_0 T_\lambda h_k$. Also $\{w_k\}$ is Cauchy in

X. If $T_\lambda h_k \to v$, $w_k \to w$, then $w \in -Av$ and $h = v + \lambda w \in R(I - \lambda A)$ by $w_k = (1/\lambda)(h_k - T_\lambda h_k)$ implies $h \leftarrow h_k = T_\lambda h_k + \lambda w_k \to v + \lambda w$, $w_k \in -A_0 T_\lambda h_k$, $-A_0 \ni (T_\lambda h_k, w_k) \to (v, w) \in -A$, thus $h = v + \lambda w \in (I - \lambda A) w \subseteq R(I - \lambda A)$. The assumption is verified, and the proof is complete. ∎

THEOREM 6. *Let the assumptions of the Main Theorem hold, and let S be the semigroup of contractions on $\overline{D(A)}$ obtained from A via the Generation Theorem* [5]. *Let $v \in \overline{D(A)}$ and $t \geqslant 0$.*

(i) *If $1 \leqslant q \leqslant \infty$ and $v|_{Q_p} \in L^q(Q_p)$ then $S(t)v|_{Q_p} \in L^q(Q_p)$ and $\| S(t)v \|_q \leqslant \| v \|_q$.*

(ii) *If $v|_{Q_p} \in L^\infty(Q_p)$, then*

$$\int_0^T \int_{Q_p} \{-|S(t)v(x) - k| f_t + \operatorname{sign}_0(S(t)v(x) - k)[\phi(S(t)v(x)) - \phi(k)] f_x\}\, dx\, dt \geqslant 0$$

for every $f(t,x) \in C_0^\infty((0,T) \times Q_p)$ such that $f \geqslant 0$ and for every $k \in \mathbb{R}$ and $T > 0$.

Proof. (i) Let $v|_{Q_p} \in L^\infty(Q_p)$. Define

$$v_n(x) = \begin{cases} n & \text{if} \quad v(x) > n \\ v(x) & \text{if} \quad v(x) \leqslant n. \end{cases}$$

Then $v_n|_{Q_p} \in L^q(Q_p)$ and $\| v_n \|_q \leqslant \| v \|_q$ and $v_n \to v$ in X. Then $T_\lambda v_n = (I - \lambda A)^{-1} v_n \to (I - \lambda A)^{-1} v$ and

$$\|(I - \lambda A)^{-1} v\|_q \leqslant \liminf_{n\to\infty} \| T_\lambda v_n \|_q \leqslant \liminf_{n\to\infty} \| v_n \|_q \leqslant \| v \|_q$$

where the second inequality is given by Theorem 5(i). Since the solution $u_\epsilon(t)$ of the problem

$$\begin{aligned} \frac{u_\epsilon(t) - u_\epsilon(t-\epsilon)}{\epsilon} - Au_\epsilon(t) \ni 0 \qquad & \text{for } t \geqslant 0 \\ u_\epsilon(t) = u_0 \qquad & \text{for } t < 0 \end{aligned}$$

is given by $u_\epsilon(t) = (I - \epsilon A)\, u_0^{-[t/\epsilon]-1}$ where $[t/\epsilon]$ is the greatest integer in t/ϵ, hence we have $\| v_\epsilon \|_q \leqslant \| v \|_q$. While $S(t)v = \lim_{\epsilon\to 0} v_\epsilon(t)$, with convergence in X, hence $\| S(t)v \|_q \leqslant \| v \|_q$.

(ii) Let $v|_{Q_p} \in L^\infty(Q_p)$ and v_ϵ satisfy

$$\begin{aligned} \frac{u_\epsilon(t) - u_\epsilon(t-\epsilon)}{\epsilon} - Au_\epsilon(t) \ni 0 \qquad & \text{for } t \geqslant 0 \\ u_\epsilon(t) = v \qquad & \text{for } t < 0. \end{aligned}$$

Then $v_\epsilon(t) = (I - \epsilon A_0)^{-[t/\epsilon]-1} v$ for $t \geqslant 0$, and $\| v_\epsilon(t)\|_q \leqslant \| v \|_q$ for $q = 1, \infty$. Let $v_\epsilon(t, x) = v_\epsilon(t)(x)$. By Definition 1, we have

$$\int_{Q_p} (\operatorname{sign}_0(v_\epsilon(t, x) - k)(\phi(v_\epsilon(t, x)) - \phi(k)) \cdot f_x(t, x) \\ - \frac{1}{\epsilon}(v_\epsilon(f - \epsilon, x) - v_\epsilon(t, x)) \operatorname{sign}_0(v_\epsilon(t, x) - k) f(t, x))\, dx \geqslant 0 \tag{28}$$

for every $k \in \mathbb{R}$ and nonnegative $f \in C_0^\infty((0, T) \times Q_p)$. Let $h_\epsilon(t, x) = (v_\epsilon(t, x) - k) \operatorname{sign}_0(v_\epsilon(t, x) - k) = | v_\epsilon(t, x) - k |$. Notice that

$$\begin{aligned}(v_\epsilon(t - \epsilon, x) &- v_\epsilon(t, x)) \operatorname{sign}_0(v_\epsilon(t, x) - k) \\ &= (v_\epsilon(t - x) - k) \operatorname{sign}_0(v_\epsilon(t, x) - k) - (v_\epsilon(t, x) - k) \operatorname{sign}_0(v_\epsilon(t, x) - k) \\ &\leqslant h_\epsilon(t - \epsilon, x) - h_\epsilon(t, x). \end{aligned} \tag{29}$$

Using (29) and integrating (28) over $0 \leqslant t \leqslant T$ yields

$$\int_0^T \int_{Q_p} \Big\{ \operatorname{sign}_0(v_\epsilon(t, x) - k)(\phi(v_\epsilon(t, x)) - \phi(k)) \cdot f_x(t, x) \\ - \frac{1}{\epsilon}(h(t - \epsilon, x) - h_\epsilon(t, x)) f(t, x) \Big\}\, dx\, dt \geqslant 0. \tag{30}$$

Now

$$\begin{aligned} \frac{1}{\epsilon} \int_0^T \int_{Q_p} &\{(h_\epsilon(t - \epsilon, x) - h_\epsilon(t, x)) f(t, x)\}\, dx\, dt \\ &= \frac{1}{\epsilon} \left(\int_0^\epsilon \int_{Q_p} h_\epsilon(t - \epsilon, x) f(t, x)\, dx\, dt - \int_{T-\epsilon}^T \int_{Q_p} h_\epsilon(t, x) f(t, x)\, dx\, dt \right) \\ &\quad + \int_\epsilon^{T-\epsilon} \int_{Q_p} h_\epsilon(t, x) \frac{1}{\epsilon}(f(t + \epsilon, x) - f(t, x))\, dx\, dt. \end{aligned}$$

The first and the second integrals vanish for ϵ small enough since $f \in C_0^\infty((0, T \times Q_p)$. The convergence, $v_\epsilon(t, x) \to S(t) v(x)$ in X, uniformly in t as $\epsilon \to 0$ implies the third term tends to $\int_0^T \int_{Q_p} | S(t) v(x) - k | f_t\, dx\, dt$ as $\epsilon \to 0$. Hence (ii) follows from letting $\epsilon \to 0$ in (30).

Thus the proof of Theorem 6 is complete. ∎

According to Theorem 6, the semigroup solution is a solution in the sense of Kružkov. In fact for the problem (DE), (IC), and (PC), assuming $u_0 \in X$ and $\phi \in C^1$, the notions of limit solution, solution in the sense of Kružkov, and good integral solution are all equivalent.

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